

# Maxima of the signless Laplacian spectral radius for planar graphs\*

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## Abstract

The signless Laplacian spectral radius of a graph is the largest eigenvalue of its signless Laplacian. In this paper, we prove that the graph  $K_2 \nabla P_{n-2}$  has the maximal signless Laplacian spectral radius among all planar graphs of order  $n \geq 456$ .

**AMS Classification:** 05C50

**Keywords:** Signless Laplacian; Spectral radius; Planar graph

## 1 Introduction

The (adjacency) spectral radius of a graph is used in many fields, including chemistry, physics and computer science [4], [10], [22]. It arises a broad research now. Spectral radius of planar graphs is of great interest in harmonic analysis, and bounds on it can be used in the design and analysis of certain Monte Carlo algorithms [19]. Spectral radius of planar graphs have applications not only in the theory of algorithms but also in theoretical physics. Boots and Royle investigated the spectral radius of planar graphs motivated by an application in geography networks [2]. This makes the research about the spectral radius of planar graphs interesting and dynamic, and many interesting results have emerged (see [19], [3], [15, 16], [20], for example). Very recently the signless Laplacian has attracted the attention of researchers. Some results on the signless Laplacian spectrum have been reported since 2005 and a new spectral theory called the  $Q$ -theory is being developed by many researchers [6]-[9], [13]. A nature question is that how about the application of the  $Q$ -theory. By computer investigations of graphs [11], E. van Dam and W. Haemers found that the signless Laplacian spectrum performs better than the adjacency spectrum or Laplacian

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spectrum in distinguishing non-isomorphic graphs. In computer science, signless Laplacian spectral radius can also be used to study the properties of a network [10], [24, 25]. Schwenk and Wilson initiated the study of the eigenvalues of planar graphs [20]. In [3], D. Cao and A. Vince conjectured that  $K_2 \nabla P_{n-2}$  has the maximum spectral radius among all planar graphs of order  $n$ , where  $\nabla$  denotes the *join* of two graphs obtained from the union of these two graphs by joining each vertex of the first graph to each vertex of the second graph. The conjuncture is still open now. With the development of the  $Q$ -theory, a natural question is that what about the maximum signless Laplacian spectral radius of planar graphs. By some comparisons in [13], it seems plausible that  $K_2 \nabla P_{n-2}$  also has the maximal signless Laplacian spectral radius among planar graphs. In this paper, we confirm that among planar graphs with order  $n \geq 456$ ,  $K_2 \nabla P_{n-2}$  has the maximal signless Laplacian spectral radius.

The layout of this paper is as follows. Section 2 gives some notations and some working lemmas. In section 3, our results are presented.

## 2 Preliminary

All graphs considered in this paper are undirected and simple, i.e. no loops or multiple edges are allowed. Denote by  $G = G[V(G), E(G)]$  a graph with vertex set  $V(G)$  and edge set  $E(G)$ .  $|V(G)| = n$  is the order.  $m(G) = |E(G)|$  is edge number. Recall that given a graph  $G$ ,  $Q(G) = D(G) + A(G)$  is called the *signless Laplacian matrix* of  $G$ , where  $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$  with  $d_i = d_G(v_i)$  being the degree of vertex  $v_i$  ( $1 \leq i \leq n$ ), and  $A(G)$  is the adjacency matrix of  $G$ . The signless Laplacian spectral radius of  $G$  is the largest eigenvalue  $q(G)$  of its signless Laplacian  $Q(G)$ . From spectral graph theory, we know that if graph  $G$  is connected, then there is a positive real eigenvector corresponding to  $q(G)$ , and the unit positive eigenvector corresponding to  $q(G)$  is always called *Perron eigenvector*. For a graph  $G$  of order  $n$ , if  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  is a positive eigenvector corresponding to  $q(G)$  satisfying that  $\sum_{i=1}^n x_i = 1$ , then  $X$  is called a *standard eigenvector* corresponding to  $q(G)$ .

Denote by  $K_n$ ,  $C_n$ ,  $P_n$  a *complete graph*, a *cycle* and a *path* of order  $n$  respectively. For a graph  $G$ , we let  $V(G)$ ,  $E(G)$  denote the vertex set and the edge set respectively. If there is no ambiguity, we use  $d(v)$  instead of  $d_G(v)$ . We use  $\delta$  or  $\delta(G)$  to denote the minimum vertex degree of a graph, use  $\Delta$  or  $\Delta(G)$  to denote the largest vertex degree of a graph, and use  $\Delta'$  or  $\Delta'(G)$  to denote the second largest vertex degree in a graph. In a graph, the notation  $v_i \sim v_j$  denotes that vertex  $v_i$  is adjacent to  $v_j$ . Denote by  $K_{s,t}$  a complete bipartite graph with one part of size  $s$  and another part of size  $t$ . In a graph  $G$  of order  $n \geq 4$ , for a vertex  $u \in V(G)$ , let  $N_G(u)$  denote the neighbor set of  $u$ , and let

$N_G[u] = \{u\} \cup N_G(u)$ .  $G(u) = G[N_G[u]]$ ,  $G^\circ(u) = G[N_G(u)]$  denote the subgraphs induced by  $N_G[u]$ ,  $N_G(u)$  respectively.

The reader is referred to [1, 14] for the facts about planar and outer-planar graphs. A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a planar graph, and such a drawing is called a planar embedding of the graph. A simple planar graph is (edge) *maximal* if no edge can be added to the graph without violating planarity. In the planar embedding of a maximal planar graph  $G$  of order  $n \geq 3$ , each face is triangle. For a planar graph  $G$ , we have  $m(G) \leq 3n - 6$  with equality if and only if it is maximal. In a maximal planar graph  $G$  of order  $n \geq 4$ ,  $\delta(G) \geq 3$ . A graph  $G$  is *outer-planar* if it has a planar embedding, called *standard embedding*, in which all vertices lie on the boundary of its outer face. A simple outer-planar graph is (edge) *maximal* if no edge can be added to the graph without violating outer-planarity. In the standard embedding of a maximal outer-planar graph  $G$  of order  $n \geq 3$ , the boundary of the outer face is a Hamiltonian cycle (a cycle contains all vertices) of  $G$ , and each of the other faces is triangle. For an outer-planar graph  $G$ , we have  $m(G) \leq 2n - 3$  with equality if and only if it is maximal. In a maximal planar graph  $G$  of order  $n \geq 4$ , for a vertex  $u \in V(G)$ ,  $G^\circ(u)$  is an outer-planar graph, and  $G(u) = u \nabla G^\circ(u)$ . From a nonmaximal planar graph  $G$ , by adding edges to  $G$ , a maximal planar graph  $G'$  can be obtained. From spectral graph theory, for a graph  $G$ , it is known that  $q(G+e) > q(G)$  if  $e \notin E(G)$ . Consequently, when we consider the maxima of the signless Laplacian spectral radius among planar graphs, it suffices to consider the maximal planar graphs directly.

Next we introduce some working lemmas.

**Lemma 2.1** [21] *Let  $u$  be a vertex of a maximal outer-planar graph on  $n \geq 2$  vertices. Then  $\sum_{v \sim u} d(v) \leq n + 3d(u) - 4$ .*

**Lemma 2.2** [17] *Let  $G$  be a graph. Then  $q(G) \leq \max_{u \in V(G)} \{d_G(u) + \frac{1}{d_G(u)} \sum_{v \sim u} d_G(v)\}$ .*

**Lemma 2.3** [8] *Let  $G$  be a connected graph containing at least one edge. Then  $q(G) \geq \Delta + 1$  with equality if and only if  $G \cong K_{1,n-1}$ .*

### 3 Main results

**Lemma 3.1** *Let  $G$  be a maximal planar graph with order  $n \geq 6$  and largest degree  $\Delta(G)$ . Then  $q(G) \leq \max_{u \in V(G)} \{d_G(u) + 2 + \frac{3n-9}{d_G(u)}\}$ . Moreover,*

- (i) *if  $\Delta(G) = n - 1$ , then  $q(G) \leq n + 4 - \frac{6}{n-1}$ ;*
- (ii) *if  $\Delta(G) = n - 2$ , then  $q(G) \leq n + 3 - \frac{3}{n-2}$ ;*
- (iii) *if  $\Delta(G) \leq n - 3$ , then  $q(G) \leq n + 2$ .*

**Proof.** For any vertex  $u \in V(G)$ , let  $N_G(u) = \{v_1, v_2, \dots, v_t\}$ ,  $V_1 = V(G) \setminus N_G[u]$ . For  $1 \leq i \leq t$ , let  $\alpha_i = d_{G^\circ(u)}(v_i)$ . Note that  $m(G^\circ(u)) = |E(G^\circ(u))| = \frac{1}{2} \sum_{i=1}^t \alpha_i$ . Between  $N_G(u)$  and  $V_1$ , there are  $3n - 6 - \frac{1}{2} \sum_{i=1}^t \alpha_i - d_G(u)$  edges. Consequently,

$$\sum_{v \sim u} d_G(v) = d_G(u) + [3n - 6 - \frac{1}{2} \sum_{i=1}^t \alpha_i - d_G(u)] + \sum_{i=1}^t \alpha_i = 3n - 6 + \frac{1}{2} \sum_{i=1}^t \alpha_i.$$

From the narration in Section 2, we know that  $G^\circ(u)$  is an outer-planar graph. Then  $m(G^\circ(u)) \leq 2d_G(u) - 3$ . As a result,  $\sum_{v \sim u} d_G(v) \leq 3n - 9 + 2d_G(u)$ , and

$$d_G(u) + \frac{1}{d_G(u)} \sum_{v \sim u} d_G(v) \leq d_G(u) + 2 + \frac{3n - 9}{d_G(u)}$$

By Lemma 2.2,  $q(G) \leq \max_{u \in V(G)} \{d_G(u) + 2 + \frac{3n - 9}{d_G(u)}\}$ .

Let  $f(x) = x + 2 + \frac{3n - 9}{x}$ . It can be checked that  $f(x)$  is convex. Note that  $3 \leq d_G(u) \leq \Delta$ . Then

$$d_G(u) + \frac{1}{d_G(u)} \sum_{v \sim u} d_G(v) \leq \max\{5 + \frac{3n - 9}{3}, \Delta + 2 + \frac{3n - 9}{\Delta}\}.$$

By Lemma 2.2, then (i)-(iii) follow. This completes the proof.  $\square$

Let  $\mathcal{H} = K_2 \nabla P_{n-2}$  for  $n \geq 4$  (see Fig. 3.1) and let  $\mathcal{H} = K_n$  for  $n = 1, 2, 3$ .

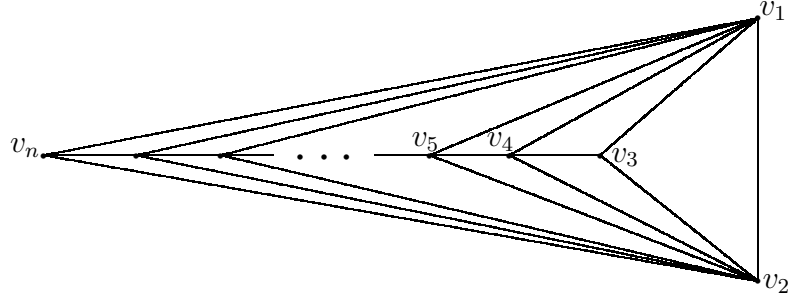


Fig 3.1.  $\mathcal{H}$

**Lemma 3.2** Suppose that the order  $n$  of  $\mathcal{H}$  is at least 5 (see Fig. 3.1). Then  $q(\mathcal{H}) > n + 2$ .

**Proof.** Let  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  be a standard eigenvector corresponding to  $q(\mathcal{H})$ , where  $x_i$  corresponds to vertex  $v_i$  and  $\sum_{i=1}^n x_i = 1$ . By symmetry,  $x_1 = x_2$ ,  $x_3 = x_n$ . By Lemma 2.3,  $q(G) \geq n$ .

Note that

$$q(G)x_1 = (n - 1)x_1 + x_2 + \sum_{i=3}^n x_i = (n - 2)x_1 + x_1 + x_2 + \sum_{i=3}^n x_i = (n - 2)x_1 + 1.$$

Then  $(q(G) - n + 2)x_1 = 1$ , and

$$x_1 = \frac{1}{q(G) - n + 2}. \quad (1)$$

Note that

$$\begin{aligned} q(G) \sum_{i=3}^n x_i &= 3x_3 + 3x_n + 4 \sum_{i=4}^{n-1} x_i + x_3 + x_n + 2 \sum_{i=4}^{n-1} x_i + (n-2)x_1 + (n-2)x_2 \\ &= 3x_3 + 3x_n + 4 \sum_{i=4}^{n-1} x_i + x_3 + x_n + 2 \sum_{i=4}^{n-1} x_i + 2(n-2)x_1 \\ &= 6 \sum_{i=3}^n x_i + 2(n-2)x_1 - 2(x_3 + x_n). \end{aligned}$$

Then

$$\begin{aligned} (q(G) - 6) \sum_{i=3}^n x_i &= 2(n-2)x_1 - 2(x_3 + x_n) = 2(n-2)x_1 - 4x_3, \\ \sum_{i=3}^n x_i &= \frac{2(n-2)x_1 - 4x_3}{q(G) - 6}, \end{aligned}$$

and

$$\begin{aligned} 1 &= \sum_{i=1}^n x_i = \frac{2(n-2)x_1 - 4x_3}{q(G) - 6} + x_1 + x_2 \\ &= \frac{2(n-2)x_1 - 4x_3}{q(G) - 6} + \frac{2}{q(G) - n + 2} \\ &= \frac{\frac{2(n-2)}{q(G) - n + 2} - 4x_3}{q(G) - 6} + \frac{2}{q(G) - n + 2}. \end{aligned}$$

Then

$$x_3 = \frac{2(n-2) - (q(G) - n)(q(G) - 6)}{4(q(G) - n + 2)}. \quad (2)$$

Note that  $q(G)x_3 = 3x_3 + x_1 + x_2 + x_4$ . Then

$$\begin{aligned} q(G)x_1 - q(G)x_3 &= (n-2)x_1 - 2x_3 + \sum_{i=5}^n x_i \\ &= (n-4)x_1 + 2x_1 - 2x_3 + \sum_{i=5}^n x_i, \end{aligned}$$

and then

$$(q(G) - 2)(x_1 - x_3) = (n-4)x_1 + \sum_{i=5}^n x_i.$$

This implies that  $x_1 > x_3$ . Combining with (1) and (2), we get

$$\frac{2(n-2) - (q(G) - n)(q(G) - 6)}{4(q(G) - n + 2)} < \frac{1}{q(G) - n + 2}. \quad (3)$$

Simplifying (3), we get  $q^2(G) - (6+n)q(G) + 4n + 8 > 0$ . Then

$$q(G) > \frac{6+n + \sqrt{(6+n)^2 - 16(n+2)}}{2} = n+2.$$

This completes the proof.  $\square$

By the narration in Section 2 and Lemmas 3.1, 3.2, to consider the maxima of the signless Laplacian spectral radius among planar graphs of order  $n \geq 5$ , it suffices to consider only the graphs with  $\Delta = n-1$  and  $\Delta = n-2$ .

**Lemma 3.3** [18] *Let  $A$  be an irreducible nonnegative square real matrix of order  $n$  and  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a real vector. For  $1 \leq i \leq n$ , let  $A_i$  be the  $i$ th row of  $A$ . If for any  $1 \leq i \leq n$ ,  $A_i X \leq r x_i$ , we say that  $AX \leq rX$ .*

**Lemma 3.4** [23] *Let  $A$  be an irreducible nonnegative square real matrix of order  $n$  and spectral radius  $\rho$ . If there exists a nonnegative real vector  $y \neq 0$  and a real coefficient polynomial function  $f$  such that  $f(A)y \leq r y$  ( $r \in \mathbf{R}$ ), then  $f(\rho) \leq r$ .*

**Lemma 3.5** *Let  $G$  be a maximal planar graph with order  $n \geq 115$  and  $d_G(v_1) = \Delta(G) = n-2$ . In  $G$ , there are exactly  $1 \leq k \leq 12$  vertices  $v_2, v_3, \dots, v_{k+1}$  such that  $\frac{n}{6} + 1 \leq d_G(v_i) \leq n-61$ , and for  $k+2 \leq i \leq n$ ,  $d_G(v_i) < \frac{n}{6} + 1$ . Then  $q(G) \leq n-2$ .*

**Proof.** Note that in  $G$ ,  $d_G(v_i) \geq 3$  for  $1 \leq i \leq n$ ,  $\sum_{i=1}^n d_G(v_i) = 2(3n-6)$  and

$$\begin{aligned} k\left(\frac{n}{6} + 1\right) + n - 2 + 3(n - k - 1) &= \left(\frac{k}{6} + 4\right)n - 2k - 5 \quad (\text{if } k \geq 13) \\ &\geq 6n + \frac{n}{6} - 31 > 6n - 12. \end{aligned}$$

Hence,  $k \leq 12$ . Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{1}{k}, & 2 \leq i \leq k+1; \\ \frac{3}{n-k-1}, & k+2 \leq i \leq n. \end{cases}$$

For  $x_1$ ,

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \leq n-2+1 + \frac{3(n-k-2)}{n-k-1} < n+2.$$

For  $x_i$  ( $k+2 \leq i \leq n$ ), when  $d(v_i) \geq k+1$ ,

$$\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} \leq d_G(v_i) + \frac{\sum_{j=1}^{k+1} x_j + \frac{3(d_G(v_i)-k-1)}{n-k-1}}{\frac{3}{n-k-1}}$$

$$\begin{aligned}
&\leq d_G(v_i) + \frac{2 + \frac{3(d_G(v_i)-k-1)}{n-k-1}}{\frac{3}{n-k-1}} \\
&\leq d_G(v_i) + \frac{2}{3}(n-k-1) + d_G(v_i) - k - 1 \\
&= 2d_G(v_i) + \frac{2}{3}n - \frac{5}{3}k - \frac{5}{3} \leq n + 2;
\end{aligned}$$

when  $d_G(v_i) \leq k$ ,

$$\begin{aligned}
\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &\leq d_G(v_i) + \frac{\sum_{j=1}^{d_G(v_i)} x_j}{\frac{3}{n-k-1}} \\
&\leq d_G(v_i) + \frac{1 + \frac{d_G(v_i)-1}{k}}{\frac{3}{n-k-1}} \\
&= d_G(v_i) + \frac{n-k-1}{3} + \frac{(d_G(v_i)-1)(n-k-1)}{3k} \\
&< k + \frac{n-k-1}{3} + \frac{n-k-1}{3} \\
&= \frac{2(n-1)+k}{3} < n+2.
\end{aligned}$$

For  $x_i$  ( $2 \leq i \leq k+1$ ), noting that  $d_G(v_i) \geq \frac{n}{6} + 1 > k$ , we get

$$\begin{aligned}
\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &= \frac{(d_G(v_i)-1)x_i + x_i + \sum_{v_j \sim v_i} x_j}{x_i} \\
&\leq d_G(v_i) - 1 + \frac{\sum_{j=1}^{k+1} x_j + \frac{3(d_G(v_i)-k)}{n-k-1}}{\frac{1}{k}} \\
&= d_G(v_i) - 1 + \frac{2 + \frac{3(d_G(v_i)-k)}{n-k-1}}{\frac{1}{k}} \\
&= (1 + \frac{3k}{n-k-1})d_G(v_i) - \frac{3k^2}{n-k-1} + 2k - 1. \quad (4)
\end{aligned}$$

Let  $f(k) = (1 + \frac{3k}{n-k-1})d_G(v_i) - \frac{3k^2}{n-k-1} + 2k - 1$ . Taking derivation of  $f(k)$  with respect to  $k$ , we get  $f'(k) = \frac{(n-k-1)(2n-2+3d_G(v_i)-8k) + 3kd_G(v_i) - 3k^2}{(n-k-1)^2}$ . Note that  $n \geq 115$ . As a result, if  $d_G(v_i) \geq k$ , then  $f'(k) > 0$ . This implies that  $f(k)$  is monotone increasing with respect to  $k$ . Note that  $d_G(v_i) \leq n-61$ . Then  $(4) < n+2$ .

Then  $Q(G)X \leq (n+2)X$ . By Lemma 3.4,  $q(G) \leq n+2$ . This completes the proof.

□

**Lemma 3.6** *Let  $G$  be a maximal planar graph with order  $n \geq 380$ ,  $d_G(v_1) = \Delta(G) = n-2$ , and  $\Delta'(G) \geq n-62$ . Then  $q(G) \leq n+2$ .*

**Proof.** Suppose  $d_G(v_2) = \Delta'(G)$ . Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ 1, & i = 2; \\ \frac{3}{n-2}, & 3 \leq i \leq n. \end{cases}$$

For  $v_1$ ,

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \leq n-2+1 + \frac{3(n-3)}{n-2} < n+2.$$

Next, there are two cases to consider.

**Case 1**  $v_2 \in N_G(v_1)$ . Suppose  $N_G(v_1) = \{v_2, v_3, \dots, v_{n-2}, v_{n-1}\}$ . Suppose  $v_1$  is in the outer face of  $G_{v_1}^\circ$ . Then  $v_n$  is in one of the inner faces of  $G_{v_1}^\circ$  (see Fig. 3.2).

For  $v_2$ ,

$$\begin{aligned} \frac{d_G(v_2)x_2 + \sum_{v_j \sim v_2} x_j}{x_2} &\leq d_G(v_2) + 1 + \frac{3(d_G(v_2) - 1)}{n-2} \\ &= d_G(v_2) + 1 + \frac{3d_G(v_2)}{n-2} - \frac{3}{n-2} \\ &\leq n+2 - \frac{3}{n-2}. \quad (d_G(v_2) \leq n-2) \end{aligned}$$

Denote by  $C_{v_1} = v_2v_3 \cdots v_{n-2}v_{n-1}v_2$  the Hamiltonian cycle in  $G_{v_1}^\circ$ . Suppose that  $v_i$ s ( $2 \leq i \leq n-1$ ) are distributed along clockwise direction on  $C_{v_1}$  and suppose  $N_{G_{v_1}^\circ}(v_2) = \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\}$ , where for  $1 \leq i \leq t-1$ ,  $2_i < 2_{i+1}$ ,  $v_{2_1} = v_3$ ,  $v_{2_t} = v_{n-1}$  (see Fig. 3.2). For  $1 \leq j \leq t$ , suppose there are  $l_{j-1}$  vertices between  $v_{2_{j-1}}$  and  $v_{2_j}$  along clockwise direction on  $C_{v_1}$ , where if  $j = 1$ , we let  $v_{2_0} = v_2$ . Along clockwise direction on  $C_{v_1}$ , suppose there are  $l_t$  vertices between  $v_{2_t}$  and  $v_2$ .

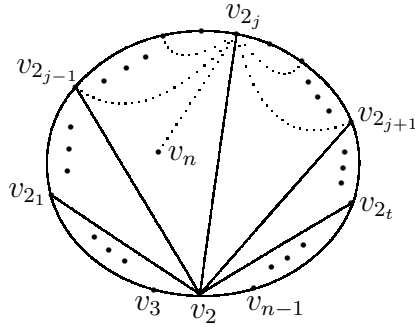


Fig. 3.2.  $d_G(v_{2_j})$

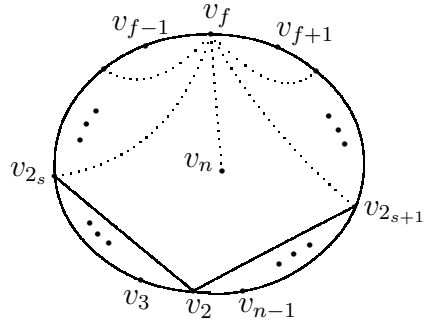


Fig. 3.3.  $d_G(v_f)$

For each  $v_{2_j}$  ( $1 \leq j \leq t$ , see Fig. 3.2), noting that

$$l_{j-1} + l_j \leq n-3 - d_G(v_2)$$

and  $d_G(v_2) \geq n-62$ , then

$$d_G(v_{2_j}) \leq l_{j-1} + l_j + 5 \leq n+2 - d_G(v_2) \leq 64,$$



and then

$$\begin{aligned}
\frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k \sim v_{2_j}} x_k}{x_{2_j}} &\leq d_G(v_{2_j}) + \frac{2 + \frac{3(d_G(v_{2_j})-2)}{n-2}}{\frac{3}{n-2}} \\
&= 2d_G(v_{2_j}) - 2 + \frac{2}{3}(n-2) \\
&\leq 126 + \frac{2}{3}(n-2) \leq n+2. \quad (n \geq 380)
\end{aligned}$$

For each  $v_f \in (N_G(v_1) \setminus \{v_2, v_{2_1}, v_{2_2}, \dots, v_{2_t}\})$ , then along clockwise direction on  $C_{v_1}$ , there exists  $0 \leq s \leq t$  such that  $v_f$  is between  $v_{2_s}$  and  $v_{2_{s+1}}$ , where  $v_{2_{t+1}} = v_2$  (see Fig. 3.3). Note that

$$l_s \leq n - 3 - d_G(v_2).$$

Then  $d_G(v_f) \leq l_s + 3 \leq n - d_G(v_2) \leq 62$ . As a result,

$$\begin{aligned}
\frac{d_G(v_f)x_f + \sum_{v_k \sim v_f} x_k}{x_f} &\leq d_G(v_f) + \frac{2 + \frac{3(d_G(v_f)-2)}{n-2}}{\frac{3}{n-2}} \\
&= 2d_G(v_f) - 2 + \frac{2}{3}(n-2) \\
&\leq 122 + \frac{2}{3}(n-2) \leq n+2. \quad (n \geq 380)
\end{aligned}$$

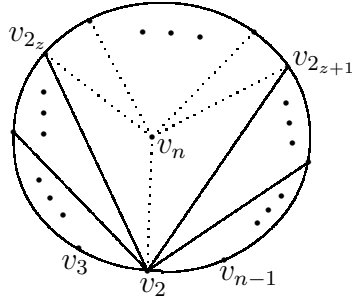


Fig. 3.4.  $d_G(v_n)$

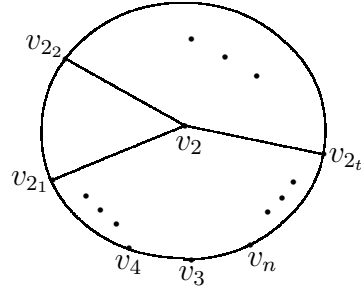


Fig. 3.5.  $d_G(v_2)$

For  $v_n$ , note that  $v_n$  is in one of the inner faces of  $G_{v_1}^\circ$ . Suppose that in  $G_{v_1}^\circ$ ,  $v_n$  is in a face  $v_2v_{2_z}v_{2_{z+1}}v_{2_{z+2}} \cdots v_{2_{z+1}}v_2$  (see Fig. 3.4). Note that  $l_z \leq n - 3 - d_G(v_2)$  and  $d_G(v_n) \leq l_z + 3$ . Then  $d_G(v_n) \leq n - d_G(v_2) \leq 62$ , and

$$\begin{aligned}
\frac{d_G(v_n)x_n + \sum_{v_k \sim v_n} x_k}{x_n} &\leq d_G(v_n) + \frac{1 + \frac{3(d_G(v_n)-1)}{n-2}}{\frac{3}{n-2}} \\
&= 2d_G(v_n) - 1 + \frac{n-2}{3} \\
&\leq 123 + \frac{1}{3}(n-2) < n+2 \quad (n \geq 380).
\end{aligned}$$

**Case 2**  $v_2 \notin N_G(v_1)$ . Suppose that  $v_1$  is in the outer face of  $G_{v_1}^\circ$ . Then  $v_2$  is in one of the inner faces of  $G_{v_1}^\circ$ . Then  $N_G(v_1) = \{v_3, v_4, v_5, \dots, v_{n-1}, v_n\}$ . Suppose that

$C_{v_1} = v_3v_4 \cdots v_{n-1}v_nv_3$  is the Hamiltonian cycle in  $G_{v_1}^\circ$ ,  $v_i$ s ( $3 \leq i \leq n$ ) are distributed along clockwise direction on  $C_{v_1}$ , and suppose  $N_{G_{v_1}^\circ}(v_2) = \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\}$ , where for  $1 \leq i \leq t-1$ ,  $2_i < 2_{i+1}$  (see Fig. 3.5). For  $2 \leq j \leq t$ , along clockwise direction on  $C_{v_1}$ , suppose there are  $l_{j-1}$  vertices between  $v_{2_{j-1}}$  and  $v_{2_j}$ . Suppose that there are  $l_t$  vertices between  $v_{2_t}$  and  $v_{2_1}$ .

For each  $v_{2_j}$  ( $2 \leq j \leq t$ ), noting that  $l_{j-1} + l_j \leq n - 2 - d_G(v_2)$  and  $d_G(v_2) \geq n - 62$ , then  $d_G(v_{2_j}) \leq l_{j-1} + l_j + 4 \leq n + 2 - d_G(v_2) \leq 64$ ; for  $v_{2_1}$ , noting that  $l_1 + l_t \leq n - 2 - d_G(v_2)$ , then  $d_G(v_{2_1}) \leq l_1 + l_t + 4 \leq n + 2 - d_G(v_2) \leq 64$ . Then for each  $v_{2_j}$  ( $1 \leq j \leq t$ ),

$$\begin{aligned} \frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k \sim v_{2_j}} x_k}{x_{2_j}} &\leq d_G(v_{2_j}) + \frac{2 + \frac{3(d_G(v_{2_j})-2)}{n-2}}{\frac{3}{n-2}} \\ &= 2d_G(v_{2_j}) - 2 + \frac{2}{3}(n-2) \\ &\leq 126 + \frac{2}{3}(n-2) \leq n+2 \quad (n \geq 380). \end{aligned}$$

For each  $v_i \in (N_G(v_1) \setminus \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\})$ , along clockwise direction on  $C_{v_1}$ ,  $v_i$  is between  $v_{2_k}$  and  $v_{2_{k+1}}$  for some  $1 \leq k \leq t-1$ , or  $v_i$  is between  $v_{2_t}$  and  $v_{2_1}$ . Then for some  $1 \leq k \leq t$ ,  $d_G(v_i) \leq l_k + 2$ . Note that  $l_k \leq n - 2 - d_G(v_2)$ . Then  $d_G(v_i) \leq n - d_G(v_2) \leq 62$ , and

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_k \sim v_i} x_k}{x_i} &\leq d_G(v_i) + \frac{1 + \frac{3(d_G(v_i)-1)}{n-2}}{\frac{3}{n-2}} \\ &= 2d_G(v_i) - 1 + \frac{1}{3}(n-2) \\ &\leq 123 + \frac{1}{3}(n-2) < n+2 \quad (n \geq 380). \end{aligned}$$

For  $v_2$ ,

$$\frac{d_G(v_2)x_2 + \sum_{v_k \sim v_2} x_k}{x_2} \leq d_G(v_2) + \frac{3d_G(v_2)}{n-2} \leq n+1. \quad (d_G(v_2) \leq n-2)$$

By above discussion, we get  $Q(G)X \leq (n+2)X$ . By Lemma 3.4, we get that  $q(G) \leq n+2$ . This completes the proof.  $\square$

**Lemma 3.7** *Let  $G$  be a maximal planar graph with order  $n \geq 4$  and  $d(v_1) = \Delta(G) = n-2$ . If for  $2 \leq i \leq n$ ,  $d_G(v_i) < 1 + \frac{n}{6}$ . Then  $q(G) \leq n-2$ .*

**Proof.** Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{4}{n-1}, & 2 \leq i \leq n. \end{cases}$$

For  $v_1$ ,

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \leq n-2 + \frac{4(n-2)}{n-1} < n+2.$$

For  $v_i$  ( $2 \leq i \leq n$ ),

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &\leq d_G(v_i) + \frac{1 + \frac{4(d_G(v_i)-1)}{n-1}}{\frac{4}{n-1}} \\ &\leq 2d_G(v_i) - 1 + \frac{n-1}{4} \\ &< \frac{7n}{12} + \frac{3}{4} < n+2. \end{aligned}$$

As a result,  $Q(G)X \leq (n+2)X$ . By Lemma 3.4, we get that  $q(G) \leq n+2$ . This completes the proof.  $\square$

**Theorem 3.8** *Let  $G$  be a maximal planar graph with order  $n \geq 380$  and  $\Delta(G) = n-2$ . Then  $q(G) \leq n+2$ .*

**Proof.** This theorem follows from Lemmas 3.5-3.7.  $\square$

**Lemma 3.9** *Let  $G$  be a maximal planar graph with order  $n \geq 91$  and  $d_G(v_1) = \Delta(G) = n-1$ . There are exactly  $1 \leq k \leq 13$  vertices  $v_2, v_3, \dots, v_{k+1}$  in  $G$  such that*

$$\frac{n}{7} + \frac{19}{7} \leq d(v_i) \leq n-75.$$

*For  $k+2 \leq i \leq n$ ,  $d_G(v_i) < \frac{n}{7} + \frac{19}{7}$ . Then  $q(G) \leq n+2$ .*

**Proof.** Note that in  $G$ ,  $d_G(v_i) \geq 3$  for  $1 \leq i \leq n$ ,  $\sum_{i=1}^n d_G(v_i) = 2(3n-6)$ , and note that if  $k \geq 14$ ,

$$k\left(\frac{n}{7} + \frac{19}{7}\right) + n-1 + 3(n-k-1) \geq 14\left(\frac{n}{7} + \frac{19}{7}\right) + n-1 + 3(n-15) > 6n-12.$$

Hence,  $k \leq 13$ . Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{2}{3k}, & 2 \leq i \leq k+1; \\ \frac{7}{3(n-k-1)}, & k+2 \leq i \leq n. \end{cases}$$

For  $v_1$ ,

$$\frac{(n-1)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} = n-1+3 = n+2.$$

For  $v_i$  ( $k+2 \leq i \leq n$ ), if  $d_G(v_i) \geq k+1$ , then

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &\leq d_G(v_i) + \frac{\frac{5}{3} + \frac{7(d_G(v_i)-k-1)}{3(n-k-1)}}{\frac{7}{3(n-k-1)}} \\ &\leq 2d_G(v_i) - k - 1 + \frac{5}{7}(n-k-1) \\ &\leq 2d_G(v_i) - \frac{12k}{7} - \frac{12}{7} + \frac{5}{7}n \leq n+2; \end{aligned}$$

if  $d_G(v_i) \leq k$ , then

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &\leq d_G(v_i) + \frac{\frac{5}{3}}{\frac{7}{3(n-k-1)}} \\ &\leq d_G(v_i) + \frac{5}{7}(n-k-1) < n+2. \end{aligned}$$

For  $v_i$  ( $2 \leq i \leq k+1$ ),  $d_G(v_i) \geq k$ , then

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &= \frac{(d_G(v_i) - 1)x_i + x_i + \sum_{v_j \sim v_i} x_j}{x_i} \\ &\leq d_G(v_i) - 1 + \frac{\sum_{j=1}^{k+1} x_j + \frac{7(d_G(v_i)-k)}{3(n-k-1)}}{\frac{2}{3k}} \\ &\leq d_G(v_i) - 1 + \frac{\frac{5}{3} + \frac{7(d_G(v_i)-k)}{3(n-k-1)}}{\frac{2}{3k}} \\ &\leq d_G(v_i) - 1 + \frac{5}{2}k + \frac{\frac{7}{2}k(d_G(v_i) - k)}{n-k-1}. \quad (5) \end{aligned}$$

As the proof of Lemma 3.5, noting that  $d_G(v_i) \leq n-75$ , we can prove that (5)  $\leq n+2$ . Then  $Q(G)X \leq (n+2)X$ . By Lemma 3.4, we get that  $q(G) \leq n+2$ . This completes the proof.  $\square$

**Lemma 3.10** *Let  $G$  be a maximal planar graph with order  $n \geq 6$  and  $d(v_1) = \Delta(G) = n-1$ . If for  $2 \leq i \leq n$ ,  $d_G(v_i) < \frac{n}{7} + \frac{19}{7}$ . Then  $q(G) \leq n+2$ .*

**Proof.** Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{3}{n-1}, & 2 \leq i \leq n. \end{cases}$$

For  $v_1$ ,

$$\frac{(n-1)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} = n-1+3 = n+2.$$

For  $v_i$  ( $2 \leq i \leq n$ ),

$$\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} \leq d_G(v_i) + \frac{1 + \frac{3(d_G(v_i)-1)}{n-1}}{\frac{3}{n-1}}$$

$$\begin{aligned}
&= 2d_G(v_i) - 1 + \frac{n-1}{3} \\
&< \frac{2n}{7} + \frac{31}{7} + \frac{n-1}{3} \\
&\leq n+2. \quad (n \geq 6)
\end{aligned}$$

As a result,  $Q(G)X \leq (n+2)X$ . By Lemma 3.4, we get that  $q(G) \leq n+2$ . This completes the proof.  $\square$

**Lemma 3.11** *Let  $G$  be a maximal planar graph with order  $n \geq 461$ ,  $d_G(v_1) = \Delta(G) = n-1$  and  $n-81 \leq \Delta'(G) \leq n-4$ . Then  $q(G) \leq n+2$ .*

**Proof.** Suppose  $d_G(v_2) = \Delta'(G)$ . Let  $X = (x_1, x_2, x_3, \dots, x_n)^T$  be a positive vector satisfying that  $x_i$  corresponds to vertex  $v_i$  and

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{4}{7}, & i = 2; \\ \frac{17}{7(n-2)}, & 3 \leq i \leq n. \end{cases}$$

For  $v_1$ ,

$$\frac{(n-1)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} = n-1+3 = n+2.$$

For  $v_2$ ,

$$\begin{aligned}
\frac{d_G(v_2)x_2 + \sum_{v_j \sim v_2} x_j}{x_2} &\leq d_G(v_2) + \frac{1 + \frac{17(d_G(v_2)-1)}{7(n-2)}}{\frac{4}{7}} \\
&= d_G(v_2) + \frac{7}{4} + \frac{17(d_G(v_2)-1)}{4(n-2)} \\
&< n+2. \quad (d_G(v_2) \leq n-4)
\end{aligned}$$

Suppose that  $v_1$  is in the outer face of  $G_{v_1}^\circ$ ,  $C_{v_1} = v_2v_3 \cdots v_{n-1}v_nv_2$  is the Hamiltonian cycle in  $G_{v_1}^\circ$ ,  $v_i$ s ( $2 \leq i \leq n$ ) are distributed along clockwise direction on  $C_{v_1}$ , and suppose  $N_{G_{v_1}^\circ}(v_2) = \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\}$ , where for  $1 \leq i \leq t-1$ ,  $2_i < 2_{i+1}$ ,  $v_{2_1} = v_3$ ,  $v_{2_t} = v_n$ . On  $C_{v_1}$ , along clockwise direction, for  $1 \leq j \leq t$ , suppose that there are  $l_{j-1}$  vertices between  $v_{2_{j-1}}$  and  $v_{2_j}$ , where if  $j = 1$ , we let  $v_{2_0} = v_2$ . Along clockwise direction on  $C_{v_1}$ , suppose that there are  $l_t$  vertices between  $v_{2_t}$  and  $v_2$ . For each  $v_{2_j}$  ( $1 \leq j \leq t$ ), noting that  $l_{j-1} + l_j \leq n-2-d_G(v_2)$  and  $d_G(v_2) \geq n-81$ , then  $d_G(v_{2_j}) \leq l_{j-1} + l_j + 4 \leq n+2-d_G(v_2) \leq 83$ , and then

$$\begin{aligned}
\frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k \sim v_{2_j}} x_k}{x_{2_j}} &\leq d_G(v_{2_j}) + \frac{\frac{11}{7} + \frac{17(d_G(v_{2_j})-2)}{7(n-2)}}{\frac{17}{7(n-2)}} \\
&= 2d_G(v_{2_j}) - 2 + \frac{11}{17}(n-2)
\end{aligned}$$

$$\begin{aligned}
&\leq 164 + \frac{11}{17}(n-2) \\
&\leq n+2. \quad (n \geq 456)
\end{aligned}$$

For each  $v_f \in (N_G(v_1) \setminus \{v_2, v_{2_1}, v_{2_2}, \dots, v_{2_t}\})$ , along clockwise direction, then there exists  $0 \leq s \leq t$  such that  $v_f$  is between  $v_{2_s}$  and  $v_{2_{s+1}}$  on  $C_{v_1}$ , and then  $d_G(v_f) \leq l_s + 2 \leq n - d_G(v_2) \leq 81$ . As a result,

$$\begin{aligned}
\frac{d_G(v_f)x_f + \sum_{v_k \sim v_f} x_k}{x_f} &\leq d_G(v_f) + \frac{\frac{11}{7} + \frac{17(d_G(v_f)-2)}{7(n-2)}}{\frac{17}{7(n-2)}} \\
&= 2d_G(v_f) - 2 + \frac{11}{17}(n-2) \\
&\leq 160 + \frac{11}{17}(n-2) \\
&\leq n+2. \quad (n \geq 456)
\end{aligned}$$

As a result,  $Q(G)X \leq (n+2)X$ . By Lemma 3.4, we get that  $q(G) \leq n+2$ . This completes the proof.  $\square$

**Lemma 3.12** *Let  $G$  be a maximal planar graph with order  $n \geq 15$ ,  $d_G(v_1) = \Delta(G) = n-1$ .*

- (i) *if  $\Delta'(G) = n-2$ , then  $q(G) < q(\mathcal{H})$ ;*
- (ii) *if  $\Delta'(G) = n-3$ , then  $q(G) < q(\mathcal{H})$ .*

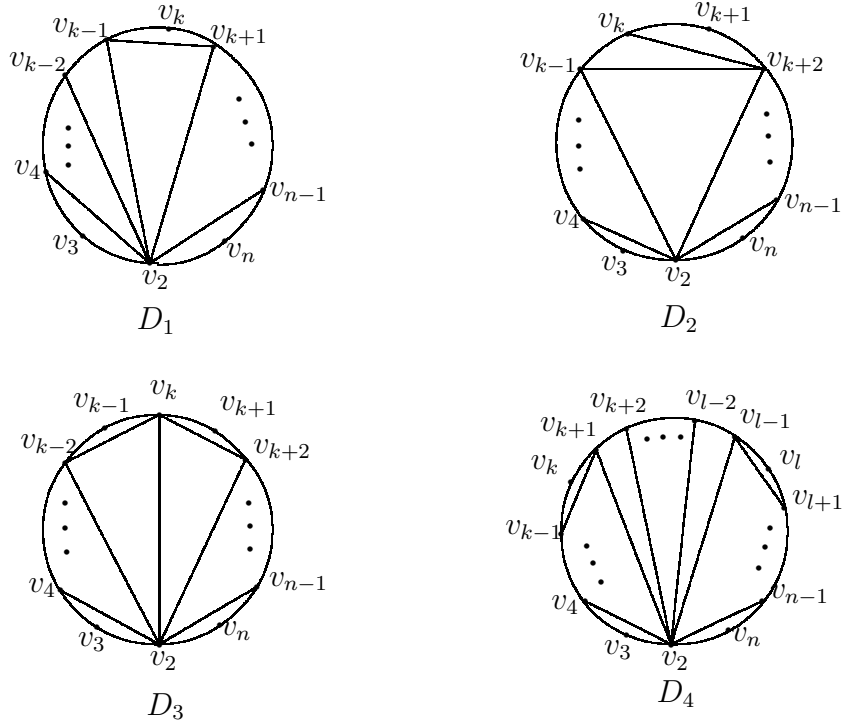


Fig. 3.6.  $D_1 - D_4$

**Proof.** Suppose  $d_G(v_2) = \Delta'(G)$ ,  $v_1$  is in the outer face of  $G_{v_1}^\circ$ , and suppose that  $C_{v_1} = v_2v_3 \cdots v_{n-1}v_nv_2$  is the Hamiltonian cycle in  $G_{v_1}^\circ$  (see Fig. 3.6).

(i) Suppose  $d_G(v_2) = \Delta'(G) = n - 2$  and  $v_k \notin N_G(v_2)$  ( $4 \leq k \leq n - 1$ ). Then  $G_{v_1}^\circ \cong D_1$  (see Fig. 3.6). For convenience, we suppose  $G_{v_1}^\circ = D_1$ . By Lemma 2.3, we get that  $q(G) > 15$ . Let  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  be the Perron eigenvector corresponding to  $q(G)$ , where  $x_i$  corresponds to vertex  $v_i$ .

Note that

$$q(G)x_k = 3x_k + x_{k-1} + x_{k+1} + x_1, \quad (6)$$

$$q(G)x_2 = (n - 2)x_2 + x_1 + \sum_{3 \leq i \leq n, i \neq k} x_i. \quad (7)$$

(6), (7) tell us that

$$\begin{aligned} q(G)x_2 - q(G)x_k &= (n - 2)x_2 - 3x_k + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+2 \leq i \leq n} x_i, \\ (q(G) - 3)(x_2 - x_k) &= (n - 5)x_2 + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+2 \leq i \leq n} x_i. \end{aligned}$$

Because  $n \geq 15$ , it follows immediately that  $x_2 > x_k$ .

Note that

$$\begin{aligned} q(G)x_{k-1} &= 5x_{k-1} + x_1 + x_2 + x_{k-2} + x_k + x_{k+1}, \\ q(G)x_{k+1} &= 5x_{k+1} + x_1 + x_2 + x_{k-1} + x_k + x_{k+2}. \end{aligned}$$

Then

$$q(G)(x_{k-1} + x_{k+1}) = 6(x_{k-1} + x_{k+1}) + 2(x_1 + x_k + x_2) + x_{k-2} + x_{k+2}. \quad (8)$$

From (6) and (7), we also get that

$$q(G)(x_2 + x_k) = (n - 2)x_2 + 3x_k + 2x_1 + 2x_{k-1} + 2x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+2 \leq i \leq n} x_i. \quad (9)$$

By (9)-(8), we get that

$$\begin{aligned} & q(G)(x_2 + x_k) - q(G)(x_{k-1} + x_{k+1}) \\ &= (n - 11)x_2 + 4(x_2 + x_k) - 4(x_{k-1} + x_{k+1}) + 3(x_2 - x_k) + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i. \end{aligned}$$

It follows that

$$(q(G) - 4)[x_2 + x_k - (x_{k-1} + x_{k+1})] = (n - 11)x_2 + 3(x_2 - x_k) + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i. \quad (10)$$

Because  $n \geq 15$ , (10) tells us that  $x_2 + x_k > x_{k-1} + x_{k+1}$ .

Let  $F = G - v_{k-1}v_{k+1} + v_2v_k$ . Note the relation between the Rayleigh quotient and the largest eigenvalue of a non-negative real symmetric matrix, and note that

$$X^T Q(F) X - X^T Q(G) X = (x_2 + x_k)^2 - (x_{k-1} + x_{k+1})^2.$$

It follows that when  $n \geq 15$ , then  $q(F) > X^T Q(F) X > X^T Q(G) X = q(G)$ . Because  $F \cong \mathcal{H}$ ,  $q(\mathcal{H}) > q(G)$ . Then (i) follows.

(ii) Suppose  $d_G(v_2) = \Delta'(G) = n - 3$ . Then there are three cases for  $G$ , that is,  $G \cong D_2$ ,  $G \cong D_3$  or  $G \cong D_4$  (see Fig. 3.6). By Lemma 2.3, we know that  $q(G) > 15$ .

**Case 1**  $G \cong D_2$ . For convenience, we suppose that  $G = D_2$ . Because  $d_G(v_2) = \Delta'(G) = n - 3$ ,  $4 \leq k \leq n - 2$ . Let  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  be the Perron eigenvector corresponding to  $q(G)$ , where  $x_i$  corresponds to vertex  $v_i$ .

Note that

$$q(G)x_{k+1} = 3x_{k+1} + x_1 + x_k + x_{k+2}, \quad (11)$$

$$q(G)x_k = 4x_k + x_1 + x_{k-1} + x_{k+1} + x_{k+2}. \quad (12)$$

By (12)-(11), we get

$$q(G)(x_k - x_{k+1}) = 3x_k - 2x_{k+1} + x_{k-1}.$$

Then

$$(q(G) - 2)(x_k - x_{k+1}) = x_k + x_{k-1}. \quad (13)$$

Because  $n \geq 15$ , (13) implies  $x_k > x_{k+1}$ .

Note that

$$q(G)x_2 = (n - 3)x_2 + x_1 + \sum_{3 \leq i \leq k-1} x_i + \sum_{k+2 \leq i \leq n} x_i. \quad (14)$$

By (14)+(12), we get

$$q(G)(x_2 + x_k) = (n - 3)x_2 + 2x_1 + 4x_k + 2x_{k-1} + x_{k+1} + 2x_{k+2} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i. \quad (15)$$

Note that

$$q(G)x_{k-1} = 5x_{k-1} + x_1 + x_2 + x_{k-2} + x_k + x_{k+2}. \quad (16)$$

By (16)+(11), we get that

$$q(G)(x_{k-1} + x_{k+1}) = 5x_{k-1} + x_{k-2} + 2x_1 + x_2 + 2x_k + 3x_{k+1} + 2x_{k+2}. \quad (17)$$

By (17)-(12), we get that

$$q(G)(x_{k-1} + x_{k+1} - x_k) = x_1 + x_2 + x_{k-2} + 4x_{k-1} + 2x_{k+1} + x_{k+2} - 2x_k.$$

Then

$$(q(G) - 2)(x_{k-1} + x_{k+1} - x_k) = x_1 + x_2 + x_{k-2} + 2x_{k-1} + x_{k+2} > 0. \quad (18)$$



Because  $n \geq 15$ , (18) implies that  $x_{k-1} + x_{k+1} > x_k$ .

By (14)-(11), we get that

$$q(G)(x_2 - x_{k+1}) = (n-3)x_2 + x_{k-1} - x_k - 3x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0.$$

Then

$$(q(G) - 4)(x_2 - x_{k+1}) = (n-7)x_2 + x_{k-1} + x_{k+1} - x_k + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0. \quad (19)$$

Because  $n \geq 15$ , (19) implies  $x_2 > x_{k+1}$ .

By (14)-(12), we get that

$$q(G)(x_2 - x_k) = (n-3)x_2 - 4x_k - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i.$$

Then

$$(q(G) - 4)(x_2 - x_k) = (n-8)x_2 + x_2 - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0. \quad (20)$$

Because  $n \geq 15$ , (20) implies that  $x_2 > x_k$ .

By (14)-(16), we get that

$$q(G)(x_2 - x_{k-1}) = (n-4)x_2 - 4x_{k-1} - x_k + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i.$$

Then

$$(q(G) - 4)(x_2 - x_{k-1}) = (n-9)x_2 + x_2 - x_k + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i. \quad (21)$$

Because  $n \geq 15$ , (21) implies that  $x_2 > x_{k-1}$ .

Note that

$$q(G)x_{k+2} = 6x_{k+2} + x_{k+1} + x_k + x_{k-1} + x_{k+3} + x_2 + x_1. \quad (22)$$

By (14)-(22), we get that

$$q(G)(x_2 - x_{k+2}) = (n-4)x_2 - 5x_{k+2} - x_k - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+4 \leq i \leq n} x_i.$$

Then

$$(q(G) - 5)(x_2 - x_{k+2}) = (n-11)x_2 + x_2 - x_k + x_2 - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+4 \leq i \leq n} x_i. \quad (23)$$

Because  $n \geq 15$ , (23) implies that  $x_2 > x_{k+2}$ .

By (16)+(22), we get that

$$q(G)(x_{k-1} + x_{k+2}) = 2x_1 + 2x_2 + x_{k-2} + 6x_{k-1} + 2x_k + x_{k+1} + 7x_{k+2} + x_{k+3}. \quad (24)$$

By (15)-(24), we get that

$$\begin{aligned} & q(G)(x_2 + x_k) - q(G)(x_{k-1} + x_{k+2}) \\ &= (n-5)x_2 - 4x_{k-1} + 2x_k - 5x_{k+2} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+4 \leq i \leq n} x_i \\ &= (n-14)x_2 + 4x_2 - 4x_{k-1} + 2x_k + 5x_2 - 5x_{k+2} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+4 \leq i \leq n} x_i. \end{aligned} \quad (25)$$

Because  $n \geq 15$ , (25) implies that  $x_2 + x_k > x_{k-1} + x_{k+2}$ .

Let  $\mathbb{F} = G - v_{k-1}v_{k+2} + v_2v_k$ . Note that  $X^T Q(\mathbb{F})X - X^T Q(G)X = (x_2 + x_k)^2 - (x_{k-1} + x_{k+2})^2$ . It follows that when  $n \geq 15$ , then  $q(\mathbb{F}) > X^T Q(\mathbb{F})X > X^T Q(G)X = q(G)$ . By (i), it follows immediately that  $q(\mathcal{H}) > q(\mathbb{F}) > q(G)$ .

**Case 2**  $G \cong D_3$ .

For convenience, we suppose that  $G = D_3$ . Because  $d_G(v_2) = \Delta'(G) = n-3$ ,  $5 \leq k \leq n-2$ . Let  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  be the Perron eigenvector corresponding to  $q(G)$ , where  $x_i$  corresponds to vertex  $v_i$ .

Note that

$$q(G)x_{k+1} = 3x_{k+1} + x_1 + x_k + x_{k+2}, \quad (26)$$

$$q(G)x_{k-1} = 3x_{k-1} + x_1 + x_k + x_{k-2}, \quad (27)$$

$$q(G)x_2 = (n-3)x_2 + x_1 + x_k + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+2 \leq i \leq n} x_i. \quad (28)$$

Then

$$\begin{aligned} q(G)x_2 - q(G)x_{k+1} &= (n-3)x_2 - 3x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i, \\ (q(G) - 3)(x_2 - x_{k+1}) &= (n-6)x_2 + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0. \end{aligned}$$

This implies that  $x_2 > x_{k+1}$ . By (28)-(27), we get that

$$q(G)x_2 - q(G)x_{k-1} = (n-3)x_2 - 3x_{k-1} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+2 \leq i \leq n} x_i.$$

Then

$$(q(G) - 3)(x_2 - x_{k-1}) = (n-6)x_2 + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+2 \leq i \leq n} x_i > 0.$$

This implies that  $x_2 > x_{k-1}$ . Note that

$$q(G)x_k = 6x_k + x_1 + x_2 + x_{k-2} + x_{k-1} + x_{k+1} + x_{k+2}, \quad (29)$$

$$q(G)x_{k+2} = 5x_{k+2} + x_1 + x_2 + x_{k+1} + x_k + x_{k+3}. \quad (30)$$

By (28)-(29), we get

$$q(G)x_2 - q(G)x_k = (n-4)x_2 - 5x_k - x_{k-1} - x_{k+1} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i.$$

Then

$$(q(G) - 5)(x_2 - x_k) = (n-11)x_2 + 2x_2 - x_{k-1} - x_{k+1} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i.$$

This implies that  $x_2 > x_k$ . By (28)-(30), we get

$$q(G)x_2 - q(G)x_{k+2} = (n-4)x_2 - 4x_{k+2} - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+4 \leq i \leq n} x_i.$$

Then

$$(q(G) - 4)(x_2 - x_{k+2}) = (n-9)x_2 + x_2 - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+4 \leq i \leq n} x_i.$$

This implies that  $x_2 > x_{k+2}$ . By (26)+(28), we get that

$$q(G)(x_2 + x_{k+1}) = (n-3)x_2 + 3x_{k+1} + 2x_1 + 2x_k + 2x_{k+2} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i. \quad (31)$$

By (29)+(30), we get that

$$q(G)(x_k + x_{k+2}) = 2x_1 + 2x_2 + 2x_{k+1} + 7x_k + 6x_{k+2} + x_{k-1} + x_{k-2} + x_{k+3}. \quad (32)$$

By (31)-(32), we get that

$$\begin{aligned} & q(G)(x_2 + x_{k+1}) - q(G)(x_k + x_{k+2}) \\ &= (n-5)x_2 + x_{k+1} - 5x_k - 4x_{k+2} - x_{k-1} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+4 \leq i \leq n} x_i \\ &= (n-15)x_2 + x_{k+1} + 5x_2 - 5x_k + 4x_2 - 4x_{k+2} + x_2 - x_{k-1} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+4 \leq i \leq n} x_i. \end{aligned} \quad (33)$$

Because  $n \geq 15$ , (33) implies that  $x_2 + x_{k+1} > x_k + x_{k+2}$ .

Let  $\mathbb{F} = G - v_k v_{k+2} + v_2 v_{k+1}$ . Note that  $X^T Q(\mathbb{F})X - X^T Q(G)X = (x_2 + x_{k+1})^2 - (x_k + x_{k+2})^2$ . It follows that when  $n \geq 15$ , then  $q(\mathbb{F}) > X^T Q(\mathbb{F})X > X^T Q(G)X = q(G)$ . Because  $n \geq 15$ , by (i), it follows immediately that  $q(\mathcal{H}) > q(\mathbb{F}) > q(G)$ .

**Case 3**  $G \cong D_4$ .

For convenience, we suppose that  $G = D_4$ . Because  $d_G(v_2) = \Delta'(G) = n-3$ ,  $4 \leq k \leq l-2$ ,  $l \leq n-1$ . Let  $X = (x_1, x_2, \dots, x_n)^T \in R^n$  be the Perron eigenvector corresponding to  $q(G)$ , where  $x_i$  corresponds to vertex  $v_i$ . Let  $\mathbb{F} = G - v_{l-1} v_{l+1} + v_2 v_l$ . As (i), it can be proved that  $q(G) < q(\mathbb{F})$ . By (i), we get that  $q(\mathbb{F}) < q(\mathcal{H})$ . Then  $q(G) < q(\mathcal{H})$ .

From above three cases, (ii) follows. This completes the proof.  $\square$

**Theorem 3.13** *Let  $G$  be a maximal planar graph with order  $n \geq 456$ ,  $d(v_1) = \Delta(G) = n - 1$  and  $\Delta'(G) \leq n - 2$ . Then  $q(G) < q(\mathcal{H})$ .*

**Proof.** This theorem follows from Lemmas 3.2 and 3.9-3.12.  $\square$

**Theorem 3.14** *Let  $G$  be a maximal planar graph with order  $n \geq 456$ . Then  $q(G) \leq q(\mathcal{H})$  with equality if and only if  $G \cong \mathcal{H}$ .*

**Proof.** This theorem follows from Lemmas 3.1, 3.2, Theorems 3.8 and 3.13.  $\square$

**Theorem 3.15** *Let  $G$  be a planar graph with order  $n \geq 456$ . Then  $q(G) \leq q(\mathcal{H})$  with equality if and only if  $G \cong \mathcal{H}$ .*

**Proof.** This theorem follows from the narrations in Section 2 and Theorem 3.14.  $\square$

**Remark** By computations with computer, we can check that among all planar graphs of order  $n \leq 6$ , the graph  $\mathcal{H}$  has the maximal signless Laplacian spectral radius. And by computation with computer, we can check and find which graph has the maximal signless Laplacian spectral radius among all planar graphs with not large order. But it seems obviously it is not a good way by computation to check and find which graph has the maximal signless Laplacian spectral radius among all planar graphs with large order because magnitudes of work need do. As for the planar graphs of order  $n \leq 455$ , we think that it need a smart way to check and find which graph has the maximal signless Laplacian spectral radius. By some computations with computer, we conjuncture that among all planar graphs of order  $n \leq 455$ , the graph  $\mathcal{H}$  still has the maximal signless Laplacian spectral radius.

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